

POISSON ALGEBRAS OF BLOCK-UPPER-TRIANGULAR BILINEAR FORMS AND BRAID GROUP ACTION

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ABSTRACT. In this paper we study a quadratic Poisson algebra structure on the space of bilinear forms on \mathbb{C}^N with the property that for any $n, m \in \mathbb{N}$ such that $nm = N$, the restriction of the Poisson algebra to the space of bilinear forms with block-upper-triangular matrix composed from blocks of size $m \times m$ is Poisson. We classify all central elements and characterise the Lie algebroid structure compatible with the Poisson algebra. We integrate this algebroid obtaining the corresponding groupoid of morphisms of block-upper-triangular bilinear forms. The groupoid elements automatically preserve the Poisson algebra. We then obtain the braid group action on the Poisson algebra as elementary generators within the groupoid. We discuss the affinisation and quantisation of this Poisson algebra, showing that in the case $m = 1$ the quantum affine algebra is the twisted q -Yangian for \mathfrak{o}_n and for $m = 2$ is the twisted q -Yangian for \mathfrak{sp}_{2n} . We describe the quantum braid group action in these two examples and conjecture the form of this action for any $m > 2$.

1. INTRODUCTION

In this paper we consider bilinear forms on \mathbb{C}^N defined by

$$\langle x, y \rangle := x^T A y, \quad \forall x, y \in \mathbb{C}^N, \quad A \in GL_N(\mathbb{C}).$$

By *block-upper-triangular bilinear form* we mean a bilinear form such that the defining matrix A is block–upper–triangular. In particular we use the following:

Notation 1.1. We let a block-upper-triangular (b.u.t.) matrix \mathbb{A} to be an $(nm) \times (nm)$ -matrix composed from blocks $\mathbb{A}_{I,J}$, $I, J = 1, \dots, n$, of size $m \times m$ with the block-upper-triangular structure: we impose the restrictions that $\mathbb{A}_{I,J} = 0$ for $I > J$ and $\det \mathbb{A}_{I,I} = 1$ for all $I = 1, \dots, n$. We denote by $\mathcal{A}_{n,m} \subset GL_{nm}$ the set of all such block-upper-triangular matrices.

In [4] it was proved that for any number of blocks n and for any size of blocks m the following brackets on the matrix elements $a_{i,j}$ of \mathbb{A} :

$$(1.1) \quad \begin{aligned} \{a_{i,j}, a_{k,l}\} &= (\text{sign}(j-l) + \text{sign}(i-k)) a_{i,l} a_{k,j} + \\ &\quad + (\text{sign}(j-k) + 1) a_{j,l} a_{i,k} + (\text{sign}(i-l) - 1) a_{l,j} a_{k,i} \end{aligned}$$

define a Poisson bracket on $\mathcal{A}_{n,m}$.

Note that the brackets (1.1) depend neither on the size of the $m \times m$ blocks nor on the number n^2 of blocks, so that the full space $GL_N(\mathbb{C})$ of non-singular $N \times N$ matrices, $N = nm$, admits this Poisson algebra (1.1). In Theorem 2.1 we show

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that the block-upper-triangular case is a Poisson reduction of the full algebra in $(GL_N(\mathbb{C}), \{\cdot, \cdot\})$.

In the case of one-dimensional blocks (i.e. upper triangular matrices with 1 on the diagonal) this algebra reduces to the Dubrovin–Ugaglia [6, 17] bracket appearing in Frobenius manifold theory and extensively studied by Bondal [1, 2]. Its quantisation is also known as Nelson–Regge algebra in 2+1-dimensional quantum gravity [14, 15], and as Fock–Rosly bracket [8] in Chern–Simons theory. We expect that for generic m this algebra may have some interesting meaning in these fields.

The affine version of the algebra (1.1) is defined in terms of the generating function:

$$(1.2) \quad \mathcal{G}_{i,j}(\lambda) := G_{i,j}^{(0)} + \sum_{p=1}^{\infty} G_{i,j}^{(p)} \lambda^{-p},$$

where $G_{i,j}^{(p)}$ denotes the entry i, j of the matrix $G^{(p)}$ and we impose that $G^{(0)} := \mathbb{A}$, our block-upper-triangular matrix, allowing the matrices $G^{(p)}$ to be arbitrary full-size matrices for $p > 0$. The Poisson brackets are postulated to be [4]

$$(1.3) \quad \begin{aligned} \{\mathcal{G}_{i,j}(\lambda), \mathcal{G}_{k,l}(\mu)\} = & \left(\text{sign}(i - k) - \frac{\lambda + \mu}{\lambda - \mu} \right) \mathcal{G}_{k,j}(\lambda) \mathcal{G}_{i,l}(\mu) + \\ & + \left(\text{sign}(j - l) + \frac{\lambda + \mu}{\lambda - \mu} \right) \mathcal{G}_{k,j}(\mu) \mathcal{G}_{i,l}(\lambda) + \\ & + \left(\text{sign}(j - k) - \frac{1 + \lambda\mu}{1 - \lambda\mu} \right) \mathcal{G}_{i,k}(\lambda) \mathcal{G}_{j,l}(\mu) + \\ & + \left(\text{sign}(i - l) + \frac{1 + \lambda\mu}{1 - \lambda\mu} \right) \mathcal{G}_{l,j}(\lambda) \mathcal{G}_{k,i}(\mu). \end{aligned}$$

We call the index p the level of the corresponding element; elements of \mathbb{A} are then called *zero-level elements*. Analogously to the case of (1.1), the algebra (1.3) is Poisson for any choice of the zero level $\mathbb{A} \in \mathcal{A}_{n,m}$, for any n, m such that $nm = N$.

In our paper [4] we related this affine extension (1.3) in the case $m = 1$ to the algebra \mathfrak{D}_n of geodesic functions on an annulus with $n \mathbb{Z}_2$ orbifold points and, simultaneously, to the algebra of monodromy data of a $n + 1$ -dimensional Frobenius manifold with one non-semi-simple point. Still in the case $m = 1$ this affine extension (1.3) turns out to be the semi-classical limit of the twisted q -Yangian for the \mathfrak{o}_n algebra introduced by Molev, Ragoucy, and Sorba [13]. A first generalisation of the above algebras to block-upper-triangular matrix case was constructed by Molev and Ragoucy in [12], where they developed the twisted Yangian $Y'_q(\mathfrak{sp}_{2n})$ for the \mathfrak{sp}_{2n} algebra. In this construction, the zero-level algebra was block-lower-triangular (equivalent to block-upper-triangular by simple transposition) with 2×2 blocks and with the restriction that each diagonal 2×2 -block have the unit determinant. In the work by Molev and Ragoucy a full description of the braid group action on $Y'_q(\mathfrak{sp}_{2n})$ was still missing and this was in part the trigger to the present work.¹

Before explaining our results on the braid group action we need to illustrate the ones on the central elements. We characterise all central elements of the Poisson algebra (1.1) and of its affine extension (1.3). They are of two types: polynomial central elements and rational central elements; together they form a set of $n[\frac{m+1}{2}] +$

¹We are particularly grateful to Alexander Molev for asking this question to us.

$\left[\frac{nm}{2}\right]$ algebraically independent central elements (here we let $[.]$ denote the integer part of a number).

In Theorem 3.2, we prove that the *polynomial central elements* for the Poisson algebra (1.1) are given by the coefficients of λ^{-k} , $k = 0, 1, \dots, \left[\frac{N+2}{2}\right]$, of the polynomial

$$\det(\mathbb{A} + \lambda^{-1}\mathbb{A}^T),$$

while for the affine Poisson algebra (1.3) they are generated by the formal series

$$\det(\mathcal{G}(\lambda)).$$

The *rational central elements* are the same for both Poisson algebras (1.1) and (1.3). They are defined by the *bottom-left minors* of the diagonal blocks of the zero level matrix \mathbb{A} , i.e. let $\mathbb{A} \in \mathcal{A}_{n,m}$, for each diagonal block $\mathbb{A}^{(I)} := \mathbb{A}_{I,I}$, $I = 1, \dots, n$ take

$$M_d^{(I)} := \det \begin{pmatrix} a_{m-d+1,1}^{(I)} & \cdots & a_{m+d-1,d}^{(I)} \\ \vdots & \ddots & \vdots \\ a_{m,1}^{(I)} & \cdots & a_{m,d}^{(I)} \end{pmatrix},$$

where $a_{i,j}^{(I)}$ denotes the i, j -th entry of $\mathbb{A}^{(I)} := \mathbb{A}_{I,I}$, then in Theorem 3.10 we prove that for every $d = 0, \dots, \left[\frac{m}{2}\right]$ and $I = 1, \dots, n$ the quantities

$$b_d^{(I)} := \frac{M_d^{(I)}}{M_{m-d}^{(I)}}$$

are central elements for both Poisson algebras (1.1) and (1.3).

Having characterised all central elements, we are ready to produce the braid group action on $\mathcal{A}_{n,m}$. For this, we follow Bondal's approach [1, 2] which consists in introducing a suitable notion of groupoid of block-upper-triangular quadratic bilinear forms in such a way that the Poisson bracket on the base space $\mathcal{A}_{n,1}$ is given in terms of the anchor map associated to the corresponding Lie algebroid. In Bondal's case, namely when $m = 1$, the Lie algebroid is isomorphic to the Lie algebroid on the cotangent bundle $T^*\mathcal{A}_{n,1}$. As soon as $m > 2$ this ceases to be true, making the integration of the Lie algebroid rather tricky. We solve this problem in Section 4 where we characterise this groupoid.²

The braid group generators are obtained as those elements in the groupoid which swap the blocks and satisfy the braid group relations. To be more precise, the braid group acting on $\mathcal{A}_{n,m}$ is B_n in which each braid $\beta_{I,I+1}$, $I = 1, \dots, n-1$ acts by changes of coordinates on \mathbb{C}^N . This action can be presented in the adjoint matrix form (see formula (5.35) below) $\beta_{I,I+1}[\mathbb{A}] = B_{I,I+1}\mathbb{A}B_{I,I+1}^T$ with the matrix $B_{I,I+1}$ having the block form (5.36).

The extended braid group transformations for the affine algebra (1.3) in the case where the zero-level matrix \mathbb{A} has the block-upper-triangular form is given by the same $\beta_{I,I+1}[\mathcal{G}(\lambda)] = B_{I,I+1}\mathcal{G}(\lambda)B_{I,I+1}^T$ and we have one extra generator $\beta_{n,1}$ given by the formulas (5.40) and (5.41).

Since the braid group elements belong to the groupoid, they preserve our algebras (1.1) and (1.3).

Finally we provide a quantisation of the affine algebra (1.3) in terms of quantum reflection equation for any m and give formulae for the quantum braid group action

²This groupoid is the natural phase space of the Poisson sigma model with target space $\mathcal{A}_{n,m}$ [3]. In this case we expect to be able to integrate the constraint equation explicitly.

in the case $m = 1$ and $m = 2$. This leads to an interesting explicit relation between the Lie algebroid of infinitesimal morphism of the b.u.t. algebra (1.1) and its R -matrix structure.

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2. POISSON REDUCTIONS OF THE ALGEBRAS (1.1) AND (1.3)

Theorem 2.1. *The affine algebra (1.3) is an abstract infinite-dimensional Poisson algebra for any choice of the block-upper-triangular form of the zero level matrix $\mathbb{A} \in \mathcal{A}_{n,m}$. Analogously, the restriction of the brackets (1.1) on $GL_N(\mathbb{C})$ to the block-upper-triangular matrices $\mathbb{A}_{n,m}$ for any $n, m \in \mathbb{N}$ such that $nm = N$, is Poisson.*

Proof. The proof of the Jacobi relations in Appendix A of [4] used only combinatorial properties and was independent on possible reductions. So, it remains only to prove the consistency of the reductions with the affine algebra (the consistency of the reductions of (1.1) is a trivial corollary). For this, let us calculate the bracket between elements of the zeroth and $k > 0$ levels. From (1.3), we have (one can obtain the formula below by taking a formal limit $\lambda \rightarrow \infty$)

$$(2.4) \quad \begin{aligned} \{a_{i,j}, G_{k,l}^{(p)}\} = & (\text{sign}(i - k) - 1)a_{k,j}G_{i,l}^{(p)} + (\text{sign}(j - l) + 1)G_{k,j}^{(p)}a_{i,l} \\ & + (\text{sign}(j - k) + 1)a_{i,k}G_{j,l}^{(p)} + (\text{sign}(i - l) - 1)a_{l,j}G_{k,i}^{(p)}. \end{aligned}$$

The right-hand side is nonzero (due to combinations of sign-factors) only for $i \leq k$ and/or $l \leq j$ and/or $k \leq j$ and/or $i \leq i$. We now use the specific form of the reduction, namely the fact that if we impose $a_{i,j} = 0$, then $a_{s,j} = 0$ for all $s \geq i$ and $a_{i,t} = 0$ for all $t \leq j$. Therefore if $i \leq k$ then $a_{k,j}$ is zero and the term $(\text{sign}(i - k) - 1)a_{k,j}G_{i,l}^{(p)}$ does not contribute. Analogously, for $k \leq j$ we have that $a_{i,k} = 0$ and the term $(\text{sign}(j - k) + 1)a_{i,k}G_{j,l}^{(p)}$ does not contribute. The same happens if $l \leq j$ and/or $i \leq i$. This proves the consistency between our reduction and the algebra (1.3). \square

Remark 2.2. A more general statement is true: let us consider block-upper-triangular matrices with blocks of different sizes, or in other words let us consider an arbitrary partition of N (previously equal to mn) into n positive integers, $N = m_1 + \dots + m_n$, and let $\mathbb{A}_{I,J}$ be a matrix of size $m_I \times m_J$. All the constructions of this paper, including the Poisson restriction (Theorem 2.1), central elements, and the action of the groupoid of b.u.t. matrices can be straightforwardly generalised to this case as well except the (classical and quantum) braid-group action, which is apparently lost in the case of different block sizes.

If we consider even more general case in which we loose the block upper triangular form, and take the Poisson reduction depicted in Fig. 1 where all elements below

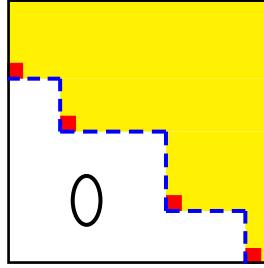


FIGURE 1. A general Poisson reduction of the algebra (1.1). All the items below the dashed broken line are zeros. The pivotal elements at the corners are marked by dark squares.

a broken line that goes as in the figure are set to be zeros, then Theorem 2.1 still remains valid, but we no longer have an algebra structure as the product of two matrices of this form does not have the same form.

2.1. Reduction to the symmetric matrices. The Poisson structure (1.1) restricts also to the space Sym_N of symmetric $N \times N$ matrices.

Proposition 2.3. *The restriction of the Poisson structure (1.1) to the space Sym_N of symmetric matrices is Poissonian.*

Proof. Let us consider the Poisson bracket of the combination $a_{i,j} - a_{j,i}$ with any element $a_{k,l}$:

$$\begin{aligned} \{a_{i,j} - a_{j,i}, a_{k,l}\} &= \text{sign}(j-l)a_{k,j}(a_{i,l} - a_{l,i}) + \text{sign}(i-k)a_{i,l}(a_{k,j} - a_{j,k}) + \\ &\quad + \text{sign}(j-k)a_{j,l}(a_{i,k} - a_{k,i}) + \text{sign}(i-l)a_{k,i}(a_{l,j} - a_{j,l}) + \\ &\quad + a_{j,l}a_{i,k} - a_{l,j}a_{k,i} - a_{i,l}a_{j,k} + a_{l,i}a_{k,j}. \end{aligned}$$

By imposing the condition $a_{r,s} = a_{s,r}$ for all r, s , the above expression is always 0. \square

The reduced bracket on Sym_N reads:

$$(2.5) \quad \{a_{i,j}, a_{k,l}\} = (\text{sign}(j-l) + \text{sign}(i-k)) a_{i,l}a_{k,j} + (\text{sign}(j-k) + \text{sign}(i-l)) a_{j,l}a_{k,i}$$

This Poisson structure on Sym_N was already studied by Bondal [2].

Remark 2.4. Observe that on the contrary the affine algebra (1.3) is not compatible with the restriction $\mathbb{A} \in Sym_N$. This can be easily seen by observing that $\{a_{i,j} - a_{j,i}, G_{k,l}^{(p)}\} \neq 0$ for $\mathbb{A} \in Sym_N$. We do not know whether an affine extension of (2.5) exists.

2.2. k -level reductions of the twisted Yangian extension.

Notation 2.5. We call the k -level reduction of the algebra (1.3) the mapping

$$(2.6) \quad \mathcal{G}(\lambda) \mapsto \mathbb{A} + \lambda^{-1}\widehat{G}^{(1)} + \cdots + \lambda^{-k+1}\widehat{G}^{(k-1)} + \lambda^{-k}\mathbb{A}^T,$$

where

$$(2.7) \quad \widehat{G}^{(i)} = \left[\widehat{G}^{(k-i)} \right]^T$$

and $\widehat{G}^{(i)} = G^{(i)}$ for $i = 1, \dots, (k-1)/2$ for odd k and for $i = 1, \dots, k/2 - 1$ for even k and $\widehat{G}_{i,j}^{(k/2)} = G_{i,j}^{(k/2)}$ for $i \geq j$ for even k , whereas the other entries of $\widehat{G}^{(i)}$ for $i \geq k/2$ are defined by the symmetry condition (2.7).

Theorem 2.6. *The mapping (2.6) defines a surjective homomorphism of the algebra (1.3) to the corresponding algebra of the elements $\mathbb{A}_{i,j}$ and $\widehat{G}_{i,j}^{(l)}$, $l = 1, \dots, k-1$, for any $k \in \mathbb{Z}_+$.*

3. CENTRAL ELEMENTS

In this section, we construct all the central elements of the algebra of block-upper-triangular matrices (1.1) and of its twisted Yangian extension (1.3). In order to find them we first need to characterise a simple automorphism and a simple anti-automorphism of the Poisson algebra (1.1).

3.1. (Anti)automorphisms of the Poisson algebra. Let $N = nm$ denote the total size of the matrix \mathbb{A} . Then the transformation

$$(3.8) \quad P[\mathbb{A}] = \tilde{\mathbb{A}}, \quad \tilde{a}_{i,j} = a_{N+1-j, N+1-i}$$

is an antiautomorphism of the Poisson algebra (1.1), that is,

$$(3.9) \quad \{\tilde{a}_{i,j}, \tilde{a}_{k,l}\} = -\{a_{i,j}, a_{k,l}\} \Big|_{a \mapsto \tilde{a}}.$$

Besides it we have the scaling transformation, which obviously leaves invariant the algebra (1.1):

$$(3.10) \quad a_{i,j} \mapsto e^{\phi_i + \phi_j} a_{i,j}, \quad \phi_i = \phi_{N+1-i},$$

where we impose the restriction on ϕ_i in order to ensure the transformation (3.10) to be consistent with the antiautomorphism (3.8). We also impose that $\sum_{i=Jm+1}^{Jm+m} \phi_i = 0$, $J = 1, \dots, n$, to ensure the preservation of the determinant condition $\det A_{J,J} = 1$ for any J .

Remark 3.1. Note that the fact that the scaling transformation (3.10) is an automorphism of the algebra (1.1) allows to restrict this algebra on the projective space \mathbb{P}^{N^2-1} . This fact is relevant due to the recent interest in the vanishing locus of quadratic Poisson algebras on projective spaces in algebraic geometry [9].

3.2. Polynomial central elements. In this subsection, we construct a part of central elements that can be obtained by standard methods based on algebra symmetries as, say, in [1] or [11].

Theorem 3.2. *The polynomial functions of the elements of the algebra (1.3) in the infinite-series expansion of $\det \mathcal{G}(\mu)$ in powers of μ^{-1} are central elements of the affine algebra (1.3).*

Proof. Although it follows from the more general statement by Molev and Ragoucy [12] on the central elements of the (quantum) Yangian algebra, we can easily verify it directly using that

$$\{\mathcal{G}_{i,j}(\lambda), \det \mathcal{G}(\mu)\} = \sum_{k,l=1}^{nm} \{\mathcal{G}_{i,j}(\lambda), \mathcal{G}_{k,l}(\mu)\} \mathcal{G}_{l,k}^{-1}(\mu)$$

(the invertibility of \mathbb{A} ensures the existence of the inverse matrix $\mathcal{G}^{-1}(\mu)$ at least as a formal series). We now substitute the bracket (1.3), and using the obvious

identities $\sum_{l=1}^{nm} \mathcal{G}_{x,l}(\mu) \mathcal{G}_{l,k}^{-1}(\mu) = \delta_{x,k}$ and $\sum_{k=1}^{nm} \mathcal{G}_{k,x}(\mu) \mathcal{G}_{l,k}^{-1}(\mu) = \delta_{l,x}$ for $x = i, j$, we obtain zero. \square

Corollary 3.3. The coefficients in the λ^{-1} -expansion of $\det(\mathbb{A} + \lambda^{-1}\mathbb{A}^T)$ are central elements of the Poisson algebra (1.1) restricted to the block-upper-triangular matrices $\mathbb{A} \in \mathcal{A}_{n,m}$ for any choice of $n, m \in \mathbb{N}$ such that $nm = N$. They form a family of $[\frac{N}{2}]$ algebraically independent central elements.

Proof. We need the statement of Theorem 2.6 for $k = 1$:

Lemma 3.4. *The mapping*

$$(3.11) \quad \mathcal{G}(\lambda) \mapsto \mathbb{A} + \lambda^{-1}\mathbb{A}^T$$

defines a surjective homomorphism of the algebra (1.3) to the algebra (1.1).

Proof. The proof of this Lemma is obtained by a direct substitution of expression (3.11) into (1.3) using the algebra (1.1). \square

Proof of Corollary 3.3. The proof that the coefficients in the λ^{-1} -expansion of $\det(\mathbb{A} + \lambda^{-1}\mathbb{A}^T)$ are central elements of the Poisson algebra (1.1) follows directly from Theorem 3.2. The fact that no more than $[\frac{N+2}{2}]$ of them are algebraically independent follows from the simple observation that

$$\det(\mathbb{A} + \lambda^{-1}\mathbb{A}^T) = \frac{c_0\lambda^N + c_1\lambda^{N-1} + \cdots + c_N}{\lambda^N},$$

where $c_{N-k} = c_k$ for all $k = 0, 1, \dots, [\frac{N}{2}]$ and $c_0 = \det(\mathbb{A}_{11}) \det(\mathbb{A}_{22}) \cdots \cdots \det(\mathbb{A}_{nn}) = 1$.

The fact that generically the coefficients $c_1, \dots, c_{[\frac{N}{2}]}$ form a family of $[\frac{N}{2}]$ algebraically independent central elements was proved in [6] for the most reduced case $m = 1$. \square

3.3. Rational central elements. The central elements in Corollary 3.3 do not exhaust all the central elements of the algebra of entries of \mathbb{A} . We also have rational central elements. To describe them we begin by considering the case of the *nonrestricted Poisson algebra* ($GL_N, \{\cdot, \cdot\}$) where $\{\cdot, \cdot\}$ is given by (1.1), and make the following

Generality assumption: All the minors M_d of size $d \times d$ located at the lower-left corner are non-zero.

Theorem 3.5. *Under the above generality assumption, the elements*

$$\det M_{N-d} / \det M_d, \quad \text{for } d = 0, \dots, \left[\frac{N-1}{2} \right],$$

are central for the affine algebra (1.3) and are algebraically independent in the nonrestricted case.

Proof. The proof is based on the following:

Lemma 3.6. *For the nonrestricted $N \times N$ matrix \mathbb{A} in our genericity assumption, denoting $a_{k,l} = G_{k,l}^{(0)}$, we have the following commutation relations:*

$$(3.12) \quad \{\det M_d, G_{k,l}^{(p)}\} = c_{k,l}^d G_{k,l}^{(p)} \det M_d \text{ for } p = 0, 1, \dots,$$

where

$$(3.13) \quad c_{k,l}^d = -\delta_{k+d>N} + \delta_{d+1>l} + \delta_{d+1>k} - \delta_{l+d>N}$$

where $\delta_{i>j} = 1$ for $i > j$ and 0 otherwise.

Proof. Let us deal with the minor M_2 first. By the Leibnitz rule, we obtain four brackets, and using relation (2.4) we obtain four terms for each bracket. Grouping together terms with the same $G_{r,s}^{(p)}$ entry we obtain

$$(3.14) \quad \begin{aligned} \left\{ M_d, G_{k,l}^{(p)} \right\} = & (\text{sign}(N-k)-1)G_{N,l}^{(p)} \begin{vmatrix} a_{N-1,1} & a_{N-1,2} \\ a_{k,1} & a_{k,2} \end{vmatrix} + \\ & + (\text{sign}(2-l)+1)G_{k,2}^{(p)} \begin{vmatrix} a_{N-1,1} & a_{N-1,l} \\ a_{N,1} & a_{N,l} \end{vmatrix} \\ & + (\text{sign}(2-k)+1)G_{2,l}^{(p)} \begin{vmatrix} a_{N-1,1} & a_{N-1,k} \\ a_{N,1} & a_{N,k} \end{vmatrix} + \\ & + (\text{sign}(N-l)-1)G_{k,N}^{(p)} \begin{vmatrix} a_{N-1,1} & a_{N-1,2} \\ a_{l,1} & a_{l,2} \end{vmatrix} + \\ & + (\text{sign}(N-1-k)-1)G_{N-1,l}^{(p)} \begin{vmatrix} a_{k,1} & a_{k,2} \\ a_{N,1} & a_{N,2} \end{vmatrix} + \\ & + (\text{sign}(1-l)+1)G_{k,1}^{(p)} \begin{vmatrix} a_{N-1,l} & a_{N-1,2} \\ a_{N,l} & a_{N,2} \end{vmatrix} + \\ & + (\text{sign}(1-k)+1)G_{1,l}^{(p)} \begin{vmatrix} a_{N-1,k} & a_{N-1,2} \\ a_{N,k} & a_{N,2} \end{vmatrix} + \\ & + (\text{sign}(N-1-l)-1)G_{k,N-1}^{(p)} \begin{vmatrix} a_{l,1} & a_{l,2} \\ a_{N,1} & a_{N,2} \end{vmatrix} \end{aligned}$$

and each term in this sum is nonzero only for one choice of either k or l . For example consider the last term on the r.h.s.: the coefficient $(\text{sign}(N-1-l)-1)$ is nonzero only for $l = N-1$ or $l = N$. However in the latter case the determinant $\begin{vmatrix} a_{l,1} & a_{l,2} \\ a_{N,1} & a_{N,2} \end{vmatrix}$ becomes zero, so we may only choose $l = N-1$. It easily follows that (3.12) and (3.13) are satisfied.

In the case of $d > 2$ the computation is very similar: the first and fifth term above are replaced by the sum of d determinants enumerated by the index $i = N-d+1, \dots, N$ and such that in each of the corresponding matrices the i th row vector is replaced by the k th row vector multiplied by $(\text{sign}(i-k)-1)G_{i,l}^{(p)}$. If $i < k$, the corresponding determinant is zero (the matrix then contains two proportional row vectors), so the only nonzero contribution occurs when $i = k$, which is possible only if $k > n-d$, and this contribution is $-G_{k,l}^{(p)} \det M_d$, which gives the first term in the r.h.s. of (3.13). Using the same reasonings we can deal with three other terms. Because $\delta_{i<0} = 1 - \delta_{i+1>0}$ we easily obtain from (3.13) that $c_{k,l}^d = c_{k,l}^{n-d}$, which completes the proof of the Lemma. \square

The proof of the fact that the elements $\det M_{N-d}/\det M_d$ for $d = 0, \dots, [\frac{N-1}{2}]$ are central then follows by the Leibnitz rule for the Poisson bracket and by observing that $c_{k,l}^d = c_{k,l}^{N-d}$ for all k, l, d , so that $\det M_d$ and $\det M_{N-d}$ have *exactly the same* commutation relations with all of $a_{k,l}$ and with all of $g_{k,l}^{(p)}$ for $p \geq 1$ in the twisted Yangian case.

That these central elements are algebraically independent was proved by Bondal [2] already for the restriction of the algebra (1.1) to Sym_N . \square

We now formulate the general algebraic independence lemma valid in the non-restricted case.

Lemma 3.7. *The set of algebraically independent central elements of the nonrestricted Poisson algebra $(GL_N, \{\cdot, \cdot\})$ where $\{\cdot, \cdot\}$ is given by (1.1), comprises the coefficients c_k of λ^{-k} , $k = 0, 1, \dots, [N/2]$, of the expansion of $\det(\mathbb{A} + \lambda^{-1}\mathbb{A}^T)$ and the rational central elements $b_l = \det M_{N-l}/\det M_l$, $l = 1, \dots, [(N-1)/2]$ provided all $\det M_l$, $l = 1, \dots, [(N-1)/2]$ are nonzero.*

Proof. We have already proved that these elements are central and that each set $\{c_k\}$ and $\{b_l\}$ is algebraically independent. Suppose we have a function

$$F(\{c_k\}, \{b_l\}) = 0 \text{ for all values of } a_{i,j}.$$

Because any transformation (3.10) is an automorphism of the algebra (1.1), choosing $\phi_l^{(i)} = s_i(\delta_{l,i+1} - \delta_{l,i})$ for $i = 1, \dots, [(N-1)/2]$ we obtain that all c_k and all b_l with $l \neq i$ remain invariant whereas $b_i \rightarrow b_i e^{2s_i}$. This means that if $F(\{c_k\}, \{b_l\}) = 0$ for some nonzero $\{b_l\}$, then $F(\{c_k\}, \{b_l e^{2\phi_l}\}) = 0$ for any choice of $\phi_l \in \mathbb{R}$. Hence, the function F is actually independent of all of b_l and we have that $F(\{c_k\}) = 0$. Because the set of $\{c_k\}$ is algebraically independent, we have that $F \equiv 0$, which completes the proof. \square

Adding $\det \mathbb{A}$, which corresponds both to the rational central element with $d = 0$ and to the polynomial central element given by the coefficient of power 0, to the set we have $\left[\frac{N+1}{2}\right]$ central elements described by Theorem 3.5 and $\left[\frac{N}{2}\right]$ central elements from Corollary 3.3, so, in the general case of a nonrestricted algebra $(GL_N, \{\cdot, \cdot\})$ where $\{\cdot, \cdot\}$ is given by (1.1), we have exactly N algebraically independent central elements.

Remark 3.8. Elementary, but lengthy calculations demonstrate that the highest Poisson leaf dimension is not less than $N^2 - N$. Here we only briefly outline the way of proving it. For this it suffices to consider the case where all $a_{i,j}$ with $i \neq j$ are ϵ -small as compared to all $a_{i,i}$ and to retain only terms of order $O(\epsilon)$ in the Poisson relations (1.1) neglecting all the terms of order ϵ^2 . Introducing then the combination $b_{i,j} = a_{i,j} - a_{j,i}$ and retaining the elements $a_{i \geq j}$ with $i \geq j$ we observe that in the limit of small ϵ , all the $b_{i,j}$ commute with all the $a_{i \geq j}$, so that the Poisson algebra splits in two sub-algebras, the $a_{i \geq j}$ -algebra and the $b_{i,j}$ -algebra.

The $b_{i,j}$ -algebra becomes the small- ϵ limit of the Dubrovin–Ugaglia or Nelson–Regge algebra (4.25) and therefore its highest Poisson leaf dimension is $\frac{N(N-1)}{2} - \left[\frac{N}{2}\right]$. The $a_{i \geq j}$ -algebra becomes the Dubrovin–Ugaglia or Nelson–Regge algebra to which we add the diagonal elements, and therefore its highest Poisson leaf dimension is $\frac{N(N+1)}{2} - \left[\frac{N+1}{2}\right]$, so the highest rank of the Poisson relations (1.1) will be not less than $N^2 - N$, as expected.

We are now going to formulate the theorem describing the rational central elements in the general case of the block-upper-triangular matrix \mathbb{A} and its possible Yangian extensions. We begin by fixing our notation.

Notation 3.9. For a block-upper-triangular matrix \mathbb{A} with n blocks of sizes $m_i \times m_i$, $i = 1, \dots, n$ on the diagonal, let $M_d^{(i)}$, $d = 0, \dots, m_i$, $i = 1, \dots, n$, be minors of

size $d \times d$ located at lower-left corners of the corresponding *diagonal blocks* of the matrix \mathbb{A} .

Theorem 3.10. *Provided $\det M_d^{(i)}$ are nonzero, all the quotients*

$$b_d^{(i)} \equiv \det M_{m_i-d}^{(i)} / \det M_d^{(i)}, \quad d = 0, \dots, [(m_i - 1)/2], \quad i = 1, \dots, n$$

are central elements of both the algebra (1.1) and its Yangian extension (1.2)–(1.3), for any choice of the zero level $\mathbb{A} \in \mathcal{A}_{n,m}$.

The central elements $b_d^{(i)}$, $d = 0, \dots, [(m_i - 1)/2]$, $i = 1, \dots, n$, and the coefficients c_r of λ^{-r} terms ($r = 1, 2, \dots$) of the expansion of $\det \mathcal{G}(\lambda)$ constitute an algebraically independent complete set of central elements of the affine algebra (1.3) whose zero level \mathbb{A} is restricted to the block-upper-triangular form $\mathbb{A} \in \mathcal{A}_{n,m}$ for any choice of n, m .

These central elements remain central for all the k -level reductions (2.6). In this case, the complete set of algebraically independent central elements comprises the same elements $b_d^{(i)}$ as above and the elements c_r with $r = 1, \dots, [(Nk)/2]$.

In particular, the maximal dimension of the Poisson leaves for the algebra (1.1) on $\mathcal{A}_{n,m}$ is

$$\frac{n(n+1)}{2}m^2 - nm - s \left[\frac{n}{2} \right], \quad s = \begin{cases} 1 & \text{for } m \text{ odd,} \\ 0 & \text{for } m \text{ even} \end{cases}$$

which is always even.

Proof. The proof of the fact that the quotients $b_d^{(i)}$ for $d = 0, \dots, [(m_i - 1)/2]$, and $i = 1, \dots, n$ are central elements is analogous to the proof of Theorem 3.5 with the only distinction that now some of the row or column vectors will be zero because of the Poissonian restrictions. The proof of algebraic independence is analogous to the proof of Lemma 3.7 in which we must generalise the automorphism (3.10) to the affine case by setting $G_{i,j}^{(p)} \mapsto G_{i,j}^{(p)} e^{\phi_i + \phi_j}$ for all $p = 0, 1, \dots$. The computation of the maximal dimension of the Poisson leaves for the algebra (1.1) on $\mathcal{A}_{n,m}$ follows from the fact that the affine space $\mathcal{A}_{n,m}$ has dimension $\frac{n(n+1)}{2}m^2 - n$ because we have $\frac{n(n-1)}{2}$ off diagonal blocks with m^2 elements each, and n diagonal blocks with $m^2 - 1$ elements each. We then need to subtract from this the number of algebraically central elements. These are $b_d^{(i)}$ for $d = 1, \dots, [(m - 1)/2]$, and $i = 1, \dots, n$, giving $\left[\frac{m-1}{2} \right] n$ algebraically independent central elements, and c_k , $k = 1, \dots, \left[\frac{mn}{2} \right]$, giving another $\left[\frac{mn}{2} \right]$ algebraically independent central elements. \square

Remark 3.11. Note that the constructed central elements are of two, very different, sorts. Those generated by $\det(\mathbb{A} + \lambda^{-1}\mathbb{A}^T)$ are invariant under the transformation (3.10) whereas, providing all $\det M_d^{(i)}$ are nonzero, we can use transformations (3.10) to set all the central elements $\det M_{m_i-d}^{(i)} / \det M_d^{(i)}$ equal to ± 1 (in the case of real parameters ϕ_s). Then, the group of (anti)automorphisms of the Poisson algebra (1.1) in the case of the block-upper-triangular matrices from Definition 1.1 is presumably generated by the braid group transformations (5.35) with $I = 1, \dots, n-1$, by the anti-automorphism P from (3.8), and, possibly, by “inner” automorphisms β_i (in terminology of Molev and Ragoucy paper [12], where these automorphisms were constructed for the case $m = 2$), $i = 1, \dots, n$, that act nontrivially only inside the blocks $\mathbb{A}_{i,i}$, $\mathbb{A}_{i,k}$ ($n \geq k > i$), and $\mathbb{A}_{l,i}$ ($i > l \geq 1$) and mutually commute. In the case of general m , these automorphisms, if exist, must also commute mutually, and

to preserve the rational central elements, we expect they to preserve the structure of minors $M_d^{(i)}$, that is, they must correspond to a sort of transposition w.r.t. the antidiagonal of the matrix $\mathbb{A}_{i,i}$. The problem of existence of these automorphisms is under investigation.

4. GROUPOID OF BLOCK UPPER TRIANGULAR BILINEAR FORMS

In this section, we follow Bondal's approach [1, 2] which consists in introducing a suitable notion of groupoid of block–upper-triangular quadratic bilinear forms in such a way that the Poisson bracket on the base space $\mathcal{A}_{n,1}$ is given in terms of the anchor map associated to the corresponding Lie algebroid. In this approach the braid group elements are then obtained as elementary generators of this groupoid and automatically preserve the Poisson structure.

In Bondal's case, namely when $m = 1$, the Lie algebroid is isomorphic to the Lie algebroid on the cotangent bundle $T^*\mathcal{A}_{n,1}$. As soon as $m > 2$ this ceases to be true, making the integration of the Lie algebroid rather tricky. We solve this problem in this Section. Let us first recall Bondal's construction.

4.1. The case of upper-triangular matrices with one on the diagonal. In this section we recall some key results from Bondal's work [1, 2], or at least our interpretation of them.

Denote by $\mathcal{A} \subset GL_n(\mathbb{C})$ the set of all upper–triangular matrices \mathbb{A} with 1 on the diagonal. The Lie group $GL_n(\mathbb{C})$ acts on \mathbb{C}^n in the usual way, thus acting on bilinear forms as

$$\forall A, B \in GL_n(\mathbb{C}), \quad A \mapsto BAB^T.$$

This action of $GL(\mathbb{C}^n)$ does not preserve \mathcal{A} , however, for any element $\mathbb{A} \in \mathcal{A}$, one can take the subset $\mathcal{M}_\mathbb{A} \subset GL(\mathbb{C}^n)$ of elements that preserve the structure of \mathbb{A} , or in other words

$$(4.15) \quad \mathcal{M}_\mathbb{A} = \{B \in GL(\mathbb{C}^n) \mid \mathbb{A} \mapsto B\mathbb{A}B^T \in \mathcal{A}\}.$$

Let $(\mathcal{A}, \mathcal{M})$ where $\mathcal{M} = \cup_{\mathbb{A} \in \mathcal{A}} \mathcal{M}_\mathbb{A}$ be the set of pairs (\mathbb{A}, B) such that $\mathbb{A} \in \mathcal{A}$ and $B \in \mathcal{M}_\mathbb{A}$. The identity morphism is defined as

$$(4.16) \quad e = (\mathbb{A}, \mathbb{1}),$$

the inverse as

$$(4.17) \quad i : (\mathbb{A}, B) \rightarrow (B\mathbb{A}B^T, B^{-1}),$$

and the partial multiplication as

$$(4.18) \quad m((B_1\mathbb{A}B_1^T, B_2), (\mathbb{A}, B_1)) = (\mathbb{A}, B_2B_1).$$

These rules define the structure of smooth algebraic groupoid on $(\mathcal{A}, \mathcal{M})$ [1]. A smooth groupoid naturally defines a Lie algebroid $(\mathcal{A}, \mathfrak{g})$, i.e. its infinitesimal version:

$$\mathfrak{g} := \cup_{\mathbb{A} \in \mathcal{A}} \mathfrak{g}_\mathbb{A}$$

where

$$\mathfrak{g}_\mathbb{A} := \{g \in \mathfrak{gl}_n(\mathbb{C}), \mid \mathbb{A} + \mathbb{A}g + g^T\mathbb{A} \in \mathcal{A}\}.$$

We denote by $D_\mathbb{A}$ the anchor map

$$(4.19) \quad \begin{aligned} D_\mathbb{A} : \quad \mathfrak{g}_\mathbb{A} &\rightarrow T_\mathbb{A}\mathcal{A} \\ g &\mapsto \mathbb{A}g + g^T\mathbb{A}. \end{aligned}$$

The Lie bracket on the space of sections $\Gamma(\mathfrak{g})$ is defined by

$$(4.20) \quad [v_1, v_2]_{\Gamma}(\mathbb{A}) := [g_1, g_2] + \sum_{i,j} \frac{\partial v_2}{\partial a_{i,j}} (D_{\mathbb{A}}(g_1))_{i,j} - \frac{\partial v_1}{\partial a_{i,j}} (D_{\mathbb{A}}(g_2))_{i,j},$$

where for $i = 1, 2$ $v_i \in \Gamma(\mathfrak{g})$ and we denote by $g_i \in \mathfrak{g}_{\mathbb{A}}$ the image of $\mathbb{A} \in \mathcal{A}$ under v_i . Here the first term in the right hand side is the usual matrix commutator.

The following Lemma is based on the fact that the tangent bundle $T\mathcal{A}$ can be identified with the space of strictly upper triangular matrices, while the cotangent bundle $T^*\mathcal{A}$ can be identified with the space of strictly lower triangular matrices by the Killing form, which is given simply by the trace in this case.

Lemma 4.1. [1] *The map*

$$(4.21) \quad \begin{aligned} P_{\mathbb{A}} : T_{\mathbb{A}}^*\mathcal{A} &\rightarrow \mathfrak{g}_{\mathbb{A}} \\ w &\mapsto P_{-,1/2}(w\mathbb{A}) - P_{+,1/2}(w^T\mathbb{A}^T), \end{aligned}$$

where $P_{\pm,1/2}$ are the projection operators:

$$(4.22) \quad P_{\pm,1/2}a_{i,j} := \frac{1 \pm \text{sign}(j-i)}{2}a_{i,j}, \quad i, j = 1, \dots, n,$$

defines an isomorphism between the Lie algebroid $(\mathfrak{g}, D_{\mathbb{A}})$ and the Lie algebroid $(T^*\mathcal{A}, D_{\mathbb{A}}P_{\mathbb{A}})$.

The Poisson bi-vector Π on \mathcal{A} is then obtained by the anchor map on the Lie algebroid $(T^*\mathcal{A}, D_{\mathbb{A}}P_{\mathbb{A}})$ (see Proposition 10.1.4 in [10]) as:

$$(4.23) \quad \begin{aligned} \Pi : T_{\mathbb{A}}^*\mathcal{A} \times T_{\mathbb{A}}^*\mathcal{A} &\rightarrow \mathcal{C}^\infty(\mathcal{A}) \\ (\omega_1, \omega_2) &\rightarrow \text{Tr}(\omega_1 D_{\mathbb{A}} P_{\mathbb{A}}(\omega_2)) \end{aligned}$$

In coordinates one can compute the Poisson bracket by

$$(4.24) \quad \{a_{i,k}, a_{j,l}\} := \frac{\partial}{\partial da_{i,k}} \wedge \frac{\partial}{\partial da_{j,l}} \text{Tr}(da_{i,k} D_{\mathbb{A}} P_{\mathbb{A}}(da_{j,l})).$$

This gives rise to the Poisson bracket on \mathcal{A} given by the Dubrovin–Ugaglia bracket [6, 17]:

$$(4.25) \quad \begin{aligned} \{a_{ik}, a_{jl}\} &= 0, \quad \text{for } i < k < j < l, \text{ and } i < j < l < k, \\ \{a_{ik}, a_{jl}\} &= 2(a_{ij}a_{kl} - a_{il}a_{kj}), \quad \text{for } i < j < k < l, \\ \{a_{ik}, a_{kl}\} &= a_{ik}a_{kl} - 2a_{il}, \quad \text{for } i < k < l, \\ \{a_{ik}, a_{jk}\} &= -a_{ik}a_{jk} + 2a_{ij}, \quad \text{for } i < j < k, \\ \{a_{ik}, a_{il}\} &= -a_{ik}a_{il} + a_{kl}, \quad \text{for } i < k < l. \end{aligned}$$

Remark 4.2. Note that the Poisson structure (4.25) is equivalent to the one given by (1.1) by plugging in the restriction $\mathbb{A} \in \mathcal{A}$ on the right-hand side.

4.2. The groupoid in the general case. The key point in our construction is based on Remark 4.2: for any n, m our algebra (1.1) is given by the same Poisson bi-vector Π as in the case $m = 1$. So, due to equation (4.23), we must retain the same Lie algebroid structure on $T^*\mathcal{A}_{n,m}$, in other words we keep the same anchor map $D_{\mathbb{A}}P_{\mathbb{A}}$. Let us be more precise.

The tangent bundle $T\mathcal{A}_{n,m}$ is now identified with the set of block–upper–triangular matrices $\delta\mathbb{A}$ such that the diagonal blocks satisfy:

$$(4.26) \quad \text{tr } \mathbb{A}_{I,I}^{-1} \delta\mathbb{A}_{I,I} = 0,$$

Analogously the cotangent bundle $T^*\mathcal{A}_{n,m}$ is now identified with the set of block-lower-triangular matrices ω such that the diagonal blocks satisfy:

$$(4.27) \quad \text{tr } \mathbb{A}_{I,I}^{-1} \omega_{I,I} = 0,$$

The map $P_{\mathbb{A}}$ is defined as above:

$$\begin{aligned} P_{\mathbb{A}} : T_{\mathbb{A}}^* \mathcal{A}_{n,m} &\rightarrow \text{Mat}_N(\mathbb{C}) \\ w &\mapsto P_{-,1/2}(w\mathbb{A}) - P_{+,1/2}(w^T \mathbb{A}^T), \end{aligned}$$

and has a non-trivial kernel now. We now define the Lie algebroid as image of this map:

$$(4.28) \quad \mathfrak{g}_{\mathbb{A}} := \text{Im}(P_{\mathbb{A}}), \quad \mathfrak{g} := \cup_{\mathbb{A} \in \mathcal{A}_{n,m}} \mathfrak{g}_{\mathbb{A}}.$$

Lemma 4.3. *The bilinear form (4.23) considered as a bilinear form on a vector space $\mathbb{C}^{(nm)^2}$ in which we substitute arbitrary (commuting) vectors $\omega_1, \omega_2 \in \mathbb{C}^{(nm)^2}$ is skew-symmetric.*

Proof. We first write the explicit expression for the bilinear form (4.23):

$$\begin{aligned} \text{Tr}[\omega_1 D_A(P_A(\omega_2))] &= \text{Tr}\left[\omega_1 \mathbb{A} P_{-,1/2}(\omega_2 \mathbb{A}) - \omega_1 \mathbb{A} P_{+,1/2}(\omega_2^T \mathbb{A}^T)\right. \\ (4.29) \quad &\quad \left.+ \mathbb{A} \omega_1 P_{+,1/2}(\mathbb{A}^T \omega_2^T) - \mathbb{A} \omega_1 P_{-,1/2}(\mathbb{A} \omega_2)\right]. \end{aligned}$$

The assertion of the lemma follows if by substituting

$$\omega_1 = \omega_2 = \omega$$

in (4.29) we obtain zero for any $\omega \in \mathbb{C}^{(nm)^2}$. Using that

$$\begin{aligned} (4.30) \quad \omega \mathbb{A} &= P_{+,1/2}(\omega \mathbb{A}) + P_{-,1/2}(\omega \mathbb{A}), \\ \mathbb{A} \omega &= P_{+,1/2}(\mathbb{A} \omega) + P_{-,1/2}(\mathbb{A} \omega), \end{aligned}$$

for the sum of the second and third terms in the r.h.s. of (4.29) we obtain under the trace sign the expression

$$\begin{aligned} &-P_{+,1/2}(\omega \mathbb{A}) P_{+,1/2}(\omega^T \mathbb{A}^T) - P_{-,1/2}(\omega \mathbb{A}) P_{+,1/2}(\omega^T \mathbb{A}^T) \\ &+ P_{+,1/2}(\mathbb{A} \omega) P_{+,1/2}(\mathbb{A}^T \omega^T) + P_{-,1/2}(\mathbb{A} \omega) P_{+,1/2}(\mathbb{A}^T \omega^T). \end{aligned}$$

Here the second term is the transposed fourth term (with the opposite sign), so the sum of these two terms vanishes under the trace sign. In the first and third terms, only the products of diagonal projections, P_d , contribute to the trace, and we obtain under the trace sign the expression

$$\begin{aligned} &-\frac{1}{4} P_d(\omega \mathbb{A}) P_d(\omega^T \mathbb{A}^T) + \frac{1}{4} P_d(\mathbb{A} \omega) P_d(\mathbb{A}^T \omega^T) \\ &= -\frac{1}{4} P_d(\omega \mathbb{A}) P_d(\omega^T \mathbb{A}^T) + \frac{1}{4} P_d(\mathbb{A}^T \omega^T) P_d(\mathbb{A} \omega) \\ &= -\frac{1}{4} P_d(\omega \mathbb{A}) P_d(\omega^T \mathbb{A}^T) + \frac{1}{4} P_d(\omega \mathbb{A}) P_d(\omega^T \mathbb{A}^T) = 0, \end{aligned}$$

where we have used that, for any two matrices X and Y , $P_d(X)P_d(Y) = P_d(Y)P_d(X)$ and $P_d(X^T) = P_d(X)$.

For the sum of the first and fourth terms in the r.h.s. of (4.29) we obtain (using the cyclicity property of the trace)

$$\begin{aligned} (4.31) \quad &\text{Tr}[\omega \mathbb{A} P_{-,1/2}(\omega \mathbb{A}) - \mathbb{A} \omega P_{-,1/2}(\mathbb{A} \omega)] \\ &= \text{Tr}[-\omega \mathbb{A} P_{+,1/2}(\omega \mathbb{A}) + \mathbb{A} \omega P_{+,1/2}(\mathbb{A} \omega)] \end{aligned}$$

$$= \frac{1}{2} \text{Tr} [\omega \mathbb{A}[P_{-,1/2} - P_{+,1/2}](\omega \mathbb{A}) - \mathbb{A}\omega[P_{-,1/2} - P_{+,1/2}](\mathbb{A}\omega)].$$

But, for any matrix X , we have that

$$\begin{aligned} \text{Tr}[X[P_- - P_+](X)] &= \text{Tr}[[P_- + P_+ + P_d](X)[P_- - P_+](X)] \\ (4.32) \quad &= \text{Tr}[P_+(X)P_-(X) - P_-(X)P_+(X)] = 0, \end{aligned}$$

and each term in the last line of (4.31) therefore vanishes, which completes the proof. \square

Theorem 4.4. *The triple $(\mathfrak{g}, D_{\mathbb{A}}, [\cdot, \cdot])$ where the anchor map $D_{\mathbb{A}}$ is given by (4.19) and the Lie bracket $[\cdot, \cdot]$ on the space of sections $\Gamma(\mathfrak{g})$ is given by (4.20) is a Lie algebroid.*

Proof. The fact that the anchor map $D_{\mathbb{A}}$ satisfies the Leibnitz rule with respect to the Lie bracket (4.20) is already proved in [2] in the case $m = 1$ and that proof extends trivially to the case of arbitrary m . We only need to prove that $\Gamma(\mathfrak{g})$ is closed, i.e. that for any two sections v_1, v_2 , there exists $\omega \in T_{\mathbb{A}}^* \mathcal{A}_{n,m}$ such that $[v_1, v_2](\mathbb{A}) = P_{\mathbb{A}}(\omega)$. Take $\omega_i \in T_{\mathbb{A}}^* \mathcal{A}_{n,m}$ such that $v_i(\mathbb{A}) = P_{\mathbb{A}}(\omega_i)$, $i = 1, 2$. The direct calculation then yields ω :

$$\begin{aligned} \omega = \varpi_{-,1} \Big[&P_{+,1/2}(\omega_2^T \mathbb{A}^T) \omega_1 - P_{+,1/2}(\omega_1^T \mathbb{A}^T) \omega_2 + P_{+,1/2}(\omega_2 \mathbb{A}) \omega_1 - P_{+,1/2}(\omega_1 \mathbb{A}) \omega_2 + \\ &+ \omega_2 P_{+,1/2}(\mathbb{A}^T \omega_1^T) - \omega_1 P_{+,1/2}(\mathbb{A}^T \omega_2^T) + \omega_1 P_{-,1/2}(\mathbb{A}\omega_2) - \omega_2 P_{-,1/2}(\mathbb{A}\omega_1) \Big], \end{aligned}$$

where we let $\varpi_{-,1}$ denote the (natural) projection on the set of (non-strictly) block-lower-triangular matrices, $\varpi_{-,1}(GL_{nm}) = \mathcal{A}_{n,m}^T$.

This concludes the proof. \square

We now integrate the Lie algebroid $(\mathfrak{g}, D_{\mathbb{A}}, [\cdot, \cdot])$ to obtain the Lie groupoid in which we will have to pick the generators of the braid group.

Theorem 4.5. *The Lie groupoid $\mathcal{M}_{n,m}$ which integrates the Lie algebroid defined by (4.28) is given by*

$$(4.33) \quad \mathcal{M}_{n,m} := U_{A_{n,m}} \mathcal{M}_{\mathbb{A}}^{(n,m)},$$

where

$$(4.34) \quad \begin{aligned} \mathcal{M}_{\mathbb{A}}^{(n,m)} := \{B \in GL(\mathbb{C}^n) \mid B \mathbb{A} B^T \in \mathcal{A}_{n,m} \text{ and the sets of central elements coincide:} \\ \{b_d^{(I)}(B \mathbb{A} B^T)\} = \{b_d^{(I)}(\mathbb{A})\}, d = 0, \dots, [\frac{m}{2}], I = 1, \dots, n\}, \end{aligned}$$

where

$$b_d^{(I)} \equiv \det M_{m-d}^{(I)} / \det M_d^{(I)},$$

are the rational central elements of our algebras (1.1) and (1.3).

Proof. Assume there exists a Lie groupoid integrating the Lie algebroid defined by (4.28). Let us prove that it must then preserve all central elements. Let f be any central element for the algebra (1.1). Its variation along any element of the groupoid is

$$\delta f = \sum_{ij} \frac{\partial f}{\partial a_{i,j}} \delta a_{i,j} = \sum_{ij} \frac{\partial f}{\partial a_{i,j}} (D_{\mathbb{A}} P_{\mathbb{A}}(\omega))_{i,j}$$

for $\omega = \delta a_{j,i}$ (thanks to the Killing form). Using the definition (4.23) of the Poisson bi-vector Π , we have that

$$\delta f = \Pi(df, \delta a_{j,i}) = \{f, a_{i,j}\},$$

where the right hand side is zero for a central element. This shows that the Lie groupoid integrating the Lie algebroid defined by (4.28) must preserve all central elements and therefore it is defined by (4.33), (4.34). \square

We conclude this Section observing that the identity morphism, the inverse and the partial multiplication for the groupoid $\mathcal{M}_{n,m}$ are still given by (4.16), (4.17) and (4.18) respectively.

Remark 4.6. We do not tackle the question whether or not there exists a smooth groupoid on $\mathcal{A}_{n,m}$ such that its Lie algebroid structure is given by $(T^*\mathcal{A}_{n,m}, D_{\mathbb{A}}P_{\mathbb{A}})$. The interested reader is invited to look at the beautiful work by Crainic and Fernandes [5].

5. BRAID-GROUP TRANSFORMATIONS

The braid-group transformations $\beta_{I,I+1}$, $I = 1, \dots, n-1$, are transformations from the groupoid (4.33), (4.34) preserving the form of the matrix \mathbb{A} , so by construction they must preserve the Poisson structure (1.1). They act of \mathbb{A} as follows:

$$(5.35) \quad \beta_{I,I+1}[\mathbb{A}] = B_{I,I+1}\mathbb{A}B_{I,I+1}^T \equiv \tilde{\mathbb{A}},$$

where the matrix $B_{I,I+1}$ has the block form

$$(5.36) \quad B_{I,I+1} = \begin{matrix} I \\ \vdots \\ I+1 \\ \vdots \end{matrix} \left[\begin{array}{cccccc} \mathbb{E} & & & & & \\ & \ddots & & & & \\ & & \mathbb{E} & & & \\ & & & \mathbb{A}_{I,I+1}^T \mathbb{A}_{I,I}^{-T} & -\mathbb{E} & \\ & & & \mathbb{A}_{I,I} \mathbb{A}_{I,I}^{-T} & \mathbb{O} & \\ & & & & & \mathbb{E} \\ & & & & & \ddots \\ & & & & & & \mathbb{E} \end{array} \right],$$

as above, \mathbb{E} and \mathbb{O} are the respective $m \times m$ unit and zero matrices.

It is straightforward to verify that the transformation (5.35) preserves the form of the matrix \mathbb{A} with

$$(5.37) \quad \begin{aligned} \widetilde{\mathbb{A}_{I,I}} &= \mathbb{A}_{I+1,I+1}, & \widetilde{\mathbb{A}_{I+1,I+1}} &= \mathbb{A}_{I,I}, & \widetilde{\mathbb{A}_{I,I+1}} &= \mathbb{A}_{I,I+1}^T \\ J < I : \widetilde{\mathbb{A}_{J,I}} &= \mathbb{A}_{J,I} \mathbb{A}_{I,I}^{-1} \mathbb{A}_{I,I+1} - \mathbb{A}_{J,I+1}, & \widetilde{\mathbb{A}_{J,I+1}} &= \mathbb{A}_{J,I} \mathbb{A}_{I,I}^{-1} \mathbb{A}_{I,I}^T, \\ J > I+1 : \widetilde{\mathbb{A}_{I,J}} &= \mathbb{A}_{I,I+1}^T \mathbb{A}_{I,I}^{-T} \mathbb{A}_{I,J} - \mathbb{A}_{I+1,J}, & \widetilde{\mathbb{A}_{I+1,J}} &= \mathbb{A}_{I,I} \mathbb{A}_{I,I}^{-T} \mathbb{A}_{I,J} \end{aligned}$$

and with all other blocks retaining their form.

We have two theorems concerning the transformations (5.35), (5.36).

Theorem 5.1. *The transformations (5.35), (5.36) are automorphisms of the Poisson structure (1.1) restricted to the block-upper-triangular matrices \mathbb{A} from Definition 1.1.*

The statement follows from that the transformation (5.35), (5.36) is a transformation from the groupoid of block-upper-triangular matrices.

Theorem 5.2. *The transformations (5.35), (5.36) satisfy the braid-group relation,*

$$(5.38) \quad \beta_{I,I+1}\beta_{I+1,I+2}\beta_{I,I+1}[\mathbb{A}] = \beta_{I+1,I+2}\beta_{I,I+1}\beta_{I+1,I+2}[\mathbb{A}], \quad I = 1, \dots, n-2.$$

We prove this theorem and the following proposition by the direct calculation.

Proposition 5.3. *We have that $(\beta_{n-1,n} \cdots \beta_{2,3} \beta_{1,2})^n [\mathbb{A}] = \widetilde{\mathbb{A}}$, where $\widetilde{\mathbb{A}_{I,J}} = (\mathbb{A}_{I,I} \mathbb{A}_{I,I}^{-T})^{n-2} \mathbb{A}_{I,J} (\mathbb{A}_{J,J}^{-1} \mathbb{A}_{J,J}^T)^{n-2}$ and, in particular, $\widetilde{\mathbb{A}_{I,I}} = \mathbb{A}_{I,I} \forall n$.*

5.1. Extension of the braid group action to the twisted Yangian case. As in the case of the standard twisted Yangian algebra (see [4]), we have the extension of the braid-group action in the case where the matrix \mathbb{A} has the original block-upper-triangular form with all blocks having the same size $m \times m$.

Proposition 5.4. *The extended braid group transformations for the algebra (1.3) in the case where the matrix \mathbb{A} has the block-upper-triangular form described in Definition 1.1 admits the following matrix representation in terms of the matrix $\mathcal{G}(\lambda)$ (1.2):*

$$(5.39) \quad \beta_{I,I+1}[\mathcal{G}(\lambda)] = B_{I,I+1}\mathcal{G}(\lambda)B_{I,I+1}^T, \quad I = 1, \dots, n-1$$

where the matrices $B_{I,I+1}$ have the form (5.36).

The action of $\beta_{n,1}$ is

$$(5.40) \quad \beta_{n,1}[\mathcal{G}(\lambda)] = B_{n,1}(\lambda)\mathcal{G}(\lambda)(B_{n,1}(\lambda^{-1}))^T,$$

where the matrix $B_{n,1}(\lambda)$ has the block form

$$(5.41) \quad B_{n,1}(\lambda) = \begin{pmatrix} \mathbb{O} & & \lambda \mathbb{A}_{n,n} \mathbb{A}_{n,n}^{-T} \\ & \mathbb{E} & \\ & & \ddots \\ & & & \mathbb{E} \\ -\lambda^{-1} \mathbb{E} & & & [\mathbb{G}_{n,1}^{(1)}]^T \mathbb{A}_{n,n}^{-T} \end{pmatrix}$$

in which $\mathbb{G}_{n,1}^{(1)}$ is the $m \times m$ block in the lower left corner of the $mn \times mn$ matrix $G^{(1)}$.

Theorem 5.5. *The transformations (5.39), (5.40) satisfy the braid-group relation,*

$$(5.42) \quad \beta_{I,I+1}\beta_{I+1,I+2}\beta_{I,I+1}[\mathcal{G}(\lambda)] = \beta_{I+1,I+2}\beta_{I,I+1}\beta_{I+1,I+2}[\mathcal{G}(\lambda)], \quad I = 1, \dots, n \mod n.$$

6. QUANTISATION

The affine algebra (1.3) is the semiclassical limit of the quantum algebra generated by the matrix elements $G_{i,j}^{(p)}$, $i, j = 1, \dots, N$, $p \in \mathbb{Z}_{\geq 0}$ subject to the defining relations:

$$(6.43) \quad R(\lambda, \mu) \overset{1}{\mathcal{G}}(\lambda) R(\lambda^{-1}, \mu)^{T_1} \overset{2}{\mathcal{G}}(\mu) = \overset{2}{\mathcal{G}}(\mu) R(\lambda^{-1}, \mu)^{T_1} \overset{1}{\mathcal{G}}(\lambda) R(\lambda, \mu)$$

where the apex T_1 indicates the transposition in space one and the R-matrix is given by

$$R(\lambda, \mu) = (\lambda - \mu) \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q^{-1}\lambda - q\mu) \sum_i E_{ii} \otimes E_{ii} +$$

$$(6.44) \quad + (q^{-1} - q)\lambda \sum_{i < j} E_{ij} \otimes E_{ji} + (q^{-1} - q)\mu \sum_{i > j} E_{ij} \otimes E_{ji}$$

it is a solution of the Yang–Baxter equation.

In coordinates the quantum algebra relations are pretty cumbersome, let us present here the formula for the level 1 reduction of the quantum algebra, or in other words, for the quantum analogue of our algebra (1.1):

$$(6.45) \quad \begin{aligned} q^{\delta_{s,j} + \delta_{i,j}} a_{i,s} a_{j,t} - q^{\delta_{s,t} + \delta_{i,t}} a_{j,t} a_{i,s} &= (q - q^{-1}) q^{\delta_{s,i}} (\delta_{t>s} - \delta_{i>j}) a_{j,s} a_{i,t} + \\ &+ (q - q^{-1}) (q^{\delta_{s,t}} \delta_{t>i} a_{j,i} a_{t,s} - q^{\delta_{i,j}} \delta_{s>j} a_{i,j} a_{s,t}) + \\ &+ (q - q^{-1})^2 \delta_{s>i} (\delta_{t>s} - \delta_{i>j}) a_{j,i} a_{s,t}, \end{aligned}$$

where $\delta_{i>j} = 1$ for $i > j$ and 0 otherwise. For $m = 2$ this quantum algebra coincides with the twisted quantised enveloping algebra $U^{tw}(\mathfrak{sp}_{2n})$ [16, 13].

The affine quantum algebra (6.43) coincides in the case of $m = 1$ ($m = 2$) with the twisted q -Yangian $Y'_q(\mathfrak{o}_n)$ ($Y'_q(\mathfrak{sp}_{2n})$) for the orthogonal (symplectic) Lie algebra introduced in [13]. For $m > 2$ this algebra has never been studied before to the best of our knowledge.

In the semiclassical limit the affine quantum algebra (6.43) gives rise to our affine algebra (1.3). We already calculated this semiclassical limit in the case $m = 1$ in [4], for general m the computation is exactly the same.

In this paper, we construct the quantum braid-group action only in the case of the \mathfrak{sp}_{2n} case, but the main features of the technique must remain unchanged for both the general $m \times m$ -b.u.t. case and for the affine algebras.

6.1. Quantum braid group action for the \mathfrak{sp}_{2n} case. In order to quantise the braid group action, we need to find the quantum analogues of the inverse, $\mathbb{A}_{I,I}^{-1}$, the transposed, $\mathbb{A}_{I,I}^T$, and the inverse-transposed, $\mathbb{A}_{I,I}^{-T}$, matrices for the diagonal blocks and also the transposed $\mathbb{A}_{I,I+1}^T$ for the off-diagonal blocks.

We first find the laws of quantum complex conjugation for all the entries $a_{i,j}$ (these formulas are valid for all n and m). The main point is that the quantum complex conjugation needs to be an automorphism of the algebra (6.43).

From formulas (6.45), assuming all the diagonal entries $a_{i,i}$, $i = 1, \dots, mn$, to be self-adjoint operators, we obtain the laws of conjugation:

$$(6.46) \quad a_{i,i}^* = a_{i,i}, \quad \begin{cases} a_{j,i}^* = qa_{j,i} \\ a_{i,j}^* = qa_{i,j} + (1 - q^2)a_{j,i} \end{cases} \quad \text{for } i < j.$$

The last formula implies that in the case where we restrict to the b.u.t. case, the lower-triangular ($i < j$) matrix entries $a_{j,i}$ not belonging to the diagonal blocks vanish, and we obtain merely that $a_{i,j}^* = qa_{i,j}$ for all the entries of the matrices $\mathbb{A}_{I,J}$ with $I < J$.

We then have the following prescription:

(1) All the transposition operations are replaced by the Hermitian conjugations.

Note that the Hermitian conjugation of the diagonal blocks is different from the Hermitian conjugation of the off-diagonal ones. As an example, we present the result of the Hermitian conjugation for the block $\mathbb{A}_{1,1}$,

$$(6.47) \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^\dagger = \begin{pmatrix} a_{11} & qa_{21} \\ qa_{12} + (1 - q^2)a_{21} & a_{22} \end{pmatrix},$$

and

$$(6.48) \quad \mathbb{A}_{I,J}^\dagger = q[\mathbb{A}_{I,J}]^T \text{ for } J > I,$$

where the transposition is understood here and hereafter in the standard matrix sense (note however that the transpose of the product of two matrices in the quantum case is not given by the reverse order product of their transposed due to the noncommutativity of their entries, say, $(\mathbb{A}_{I,J}\mathbb{A}_{K,L})^T \neq \mathbb{A}_{K,L}^T\mathbb{A}_{I,J}^T$).

(2) The inverse $\mathbb{A}_{I,I}^{-1}$ is to be found from the operatorial identity $\mathbb{A}_{I,I}\mathbb{A}_{I,I}^{-1} = \mathbb{E}$, in which the order in which we multiply operators follows from that of the matrix multiplication. In the $m = 2$ case, the result is (we present it for $\mathbb{A}_{1,1}$, the generalisation to other 2×2 -blocks $\mathbb{A}_{I,I}$ is obvious)

$$(6.49) \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{a_{11}a_{22} - q^2a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} + (q - q^{-1})a_{21} \\ -q^2a_{21} & a_{11} \end{pmatrix}$$

Here the combination $a_{11}a_{22} - q^2a_{12}a_{21}$ is the quantum determinant of the block $\mathbb{A}_{1,1}$; all these determinants are self-adjoint central elements of the quantum algebra \mathfrak{sp}_{2n} .

(3) eventually, the inverse-transposed in the quantum case becomes $[\mathbb{A}_{I,I}^{-1}]^\dagger = [\mathbb{A}_{I,I}^\dagger]^{-1}$ where we use the expressions (6.49) and (6.47); the result is merely (we always assume that $q := e^{-i\pi\hbar}$ and, therefore, $q^* = q^{-1}$)

$$(6.50) \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-\dagger} = \frac{1}{a_{11}a_{22} - q^2a_{12}a_{21}} \begin{pmatrix} a_{22} & -q^{-1}a_{21} \\ -qa_{12} & a_{11} \end{pmatrix}$$

Theorem 6.1. *For the b.u.t matrix \mathbb{A}^\hbar from Definition 1.1 whose entries are operators subject to the conjugation law (6.46) the action of the quantum braid group has the adjoint matrix form*

$$(6.51) \quad \beta_{I,I+1}^\hbar[\mathbb{A}^\hbar] = B_{I,I+1}^\hbar \mathbb{A}^\hbar [B_{I,I+1}^\hbar]^\dagger \equiv \widetilde{\mathbb{A}}^\hbar,$$

where the matrix $B_{I,I+1}^\hbar$ has the block form (here and hereafter we assume that all matrix entries are operators and that these operators are multiplied in the order prescribed by the matrix multiplication)

$$(6.52) \quad B_{I,I+1}^\hbar = \frac{I}{I+1} \begin{pmatrix} \mathbb{E} & & & & \\ & \ddots & & & \\ & & \mathbb{E} & & \\ & & & q^{-a}\mathbb{A}_{I,I+1}^\dagger\mathbb{A}_{I,I}^{-\dagger} & -q^{-a}\mathbb{E} \\ & & & q^{-b}\mathbb{A}_{I,I}\mathbb{A}_{I,I}^{-\dagger} & \mathbb{O} \\ & & & & \mathbb{E} \\ & & & & \ddots \\ & & & & \mathbb{E} \end{pmatrix},$$

where $a = b + 1$ for all m and $b = 0$ for $m = 1$ and $b = 1$ for $m = 2$.

Before proving the theorem, we formulate the following lemma, which can be verified by a simple calculation.

Lemma 6.2. *The two quantum braid-group relations*

$$(6.53) \quad \beta_{I,I+1}^\hbar \beta_{I-1,I}^\hbar \beta_{I,I+1}^\hbar [\mathbb{A}^\hbar] = \beta_{I-1,I}^\hbar \beta_{I,I+1}^\hbar \beta_{I-1,I}^\hbar [\mathbb{A}^\hbar], \quad I = 2, \dots, n-1,$$

and

$$(6.54) \quad (\beta_{n-1,n}^\hbar \beta_{n-2,n-1}^\hbar \cdots \beta_{2,3}^\hbar \beta_{1,2}^\hbar)^n [\mathbb{A}_{I,J}] = (\mathbb{A}_{I,I} \mathbb{A}_{I,I}^{-\dagger})^{n-2} \mathbb{A}_{I,J} (\mathbb{A}_{J,J}^{-1} \mathbb{A}_{J,J}^\dagger)^{n-2},$$

which are the quantum analogues of the respective relation (5.38) and Proposition 5.3, are satisfied for any choice of the parameters a and b in the formula (6.52).

Proof. of Theorem 6.1. We need to prove that the quantum algebra (6.45) is preserved under the quantum action (6.51). It is enough to restrict to the case of $n = 3$ and concentrate on the action of $\beta_{1,2}^\hbar$:

$$\beta_{1,2}^\hbar(\mathbb{A}^\hbar) = \tilde{\mathbb{A}}^\hbar = \begin{pmatrix} \mathbb{A}_{2,2} & q^{-a+b} \mathbb{A}_{1,2}^\dagger & q^{-a} (\mathbb{A}_{1,2}^\dagger \mathbb{A}_{1,1}^{-1} \mathbb{A}_{1,3} - \mathbb{A}_{2,3}) \\ \mathbb{O} & \mathbb{A}_{1,1} & q^{-b} (\mathbb{A}_{1,1} \mathbb{A}_{1,1}^{-\dagger} \mathbb{A}_{1,3}) \\ \mathbb{O} & \mathbb{O} & \mathbb{A}_{3,3} \end{pmatrix}.$$

In order for the $(1, 2)$ matrix entry to have the same conjugation law (6.48) as the original operator $\mathbb{A}_{1,2}$ we need that $-a + b = -1$, or $a = b + 1$ for all m . However, when considering the $(2, 3)$ matrix entry, we observe the explicit difference between cases where $m = 1$ and $m = 2$. If $m = 1$, then $\mathbb{A}_{1,1} = 1$, and we just have $q^{-b} \mathbb{A}_{1,3}$, so, in order to preserve the conjugation law we must merely set $b = 0$. But in the case of 2×2 -matrices the situation becomes different: we need a rather nontrivial calculation, which involves repeated applications of the commutation relations (6.45), to demonstrate that

$$(\mathbb{A}_{1,1} \mathbb{A}_{1,1}^{-\dagger} \mathbb{A}_{1,3})^\dagger = q^{-1} [\mathbb{A}_{1,1} \mathbb{A}_{1,1}^{-\dagger} \mathbb{A}_{1,3}]^T \text{ for } m = 2.$$

So, we need to set $b = 1$ for $m = 2$ in order to preserve the conjugation law.

The proof of preserving the algebra is then a straightforward brute force computation in which one needs to check relations (6.45) entry by entry. \square

Remark 6.3. The transformation $\beta_{1,2}$ given by the formula (6.51) in the case of 2×2 -block matrix $\mathbb{A} = \begin{pmatrix} \mathbb{A}_{1,1} & \mathbb{A}_{1,2} \\ \mathbb{O} & \mathbb{A}_{2,2} \end{pmatrix}$ results in that $\mathbb{A}_{1,1} \rightarrow \mathbb{A}_{2,2}$, $\mathbb{A}_{2,2} \rightarrow \mathbb{A}_{1,1}$, and $\mathbb{A}_{1,2} \rightarrow [\mathbb{A}_{1,2}]^T$. It is easy to see from the algebra (6.43) that this transformation is an automorphism of the corresponding quantum algebra. This automorphism was stated as Theorem 4.8 in [12].

Conjecture 6.4. It is plausible that the braid-group action in the general case of $m \times m$ -matrix blocks has the same form (6.51) and (6.52) with $a = b + 1$ and $b = m - 1$ where we have to determine the inverse matrix $\mathbb{A}_{I,I}^{-1}$ from the operatorial equality $\mathbb{A}_{I,I} \mathbb{A}_{I,I}^{-1} = \mathbb{E}$ and use the conjugation rules (6.46).

Conjecture 6.5. In the twisted Yangian case (6.43) for $m = 2$ we expect to extend the same quantum conjugation relations to all levels and to quantise the element $\beta_{n,1}$ defined by (5.41) to

$$\beta_{n,1}^\hbar[\mathbb{A}^\hbar] = B_{n,1}^\hbar(\lambda) \mathbb{A}^\hbar \left[B_{n,1}^\hbar(\lambda^{-1}) \right]^\dagger,$$

where the matrix $B_{n,1}^{\hbar}(\lambda)$ has the block form:

$$B_{n,1}^{\hbar}(\lambda) = \begin{pmatrix} \mathbb{O} & & \lambda q^{-b} \mathbb{A}_{n,n} \mathbb{A}_{n,n}^{-\dagger} \\ & \mathbb{E} & \\ & & \ddots \\ & & & \mathbb{E} \\ -\lambda^{-1} q^{-b-1} \mathbb{E} & & & q^{-b-1} [\mathbb{G}_{n,1}^{(1)}]^{\dagger} \mathbb{A}_{n,n}^{-\dagger} \end{pmatrix}$$

with $b = m - 1$.

6.2. R -matrix role in the classical case. The classical Poisson bracket (1.1) was obtained by imposing

$$(6.55) \quad \{a_{i,j}, a_{k,l}\} = \frac{\partial}{\partial da_{i,j}} \wedge \frac{\partial}{\partial da_{k,l}} S,$$

where S is the Poisson bi-vector computed on the one forms $da_{i,j}$ and $da_{k,l}$:

$$S = \text{Tr}(da_{i,j} D_{\mathbb{A}} P_{\mathbb{A}}(da_{k,l})).$$

In this subsection we want to show that (6.55) can also be written in R -matrix notation as:

$$(6.56) \quad \{\mathbb{A}^1, \mathbb{A}^2\} = - \left(\mathbb{A}^1 \mathbb{A}^2 r - r \mathbb{A}^1 \mathbb{A}^2 + \mathbb{A}^2 r^{T_1} \mathbb{A}^1 - \mathbb{A}^1 r^{T_1} \mathbb{A}^2 \right),$$

where r is given by

$$r = \sum_i E_{ii} \otimes E_{ii} + 2 \sum_{i>j} E_{ij} \otimes E_{ji}.$$

Of course, we could just do a brute force computation to simply prove that the two formulae coincide. However, we prefer to show a more general proof which relies on the relation between the $P_{\mathbb{A}}$ operator and the classical R -matrix.

Given any matrix X , let us define the following four operators r_{\pm}^1, r_{\pm}^2 :

$$(6.57) \quad \begin{aligned} r_+^1(X) &:= \frac{1}{2} \text{Tr} \left(r^T X^2 \right), & r_+^2(X) &:= \frac{1}{2} \text{Tr} \left(r X^1 \right), \\ r_-^1(X) &:= \frac{1}{2} \text{Tr} \left(r X^2 \right), & r_-^2(X) &:= \frac{1}{2} \text{Tr} \left(r^T X^1 \right), \end{aligned}$$

then our matrix $g := P_{\mathbb{A}}(w)$ is given by two different formulae according to which space we want it in:

$$(6.58) \quad g = r_-^1 \left(w \mathbb{A}^2 \right) - r_+^1 \left(w^T \mathbb{A}^2 T \right),$$

$$(6.59) \quad g = r_-^2 \left(w \mathbb{A}^1 \right) - r_+^2 \left(w^T \mathbb{A}^1 T \right),$$

where r^T is the transposition in both spaces.

Now, it is clear that in R -matrix notation S is given by

$$\begin{aligned} S &= \text{Tr}[(d\mathbb{A})^T \mathbb{A} r_- [(d\mathbb{A})^T \mathbb{A}] - (d\mathbb{A})^T \mathbb{A} r_+ [d\mathbb{A} \mathbb{A}^T] \\ &\quad + \mathbb{A} (d\mathbb{A})^T r_+ [\mathbb{A}^T d\mathbb{A}] - \mathbb{A} (d\mathbb{A})^T r_- [\mathbb{A} (d\mathbb{A})^T]] \end{aligned}$$

where we did not specify the spaces as any choice leads to the same result (up to sign). For example choosing space 2 we get:

$$(6.60) \quad S = \frac{1}{2} \text{Tr} [(d\mathbb{A})^T \overset{2}{\mathbb{A}} \overset{1}{\mathbb{A}} r^T (d\mathbb{A})^T - (d\mathbb{A})^T r^T \overset{2}{\mathbb{A}} \overset{1}{\mathbb{A}} (d\mathbb{A})^T - (d\mathbb{A})^T \overset{2}{\mathbb{A}} r^{T_1} \overset{1}{\mathbb{A}} (d\mathbb{A})^T + (d\mathbb{A})^T \overset{1}{\mathbb{A}} r^{T_1} \overset{2}{\mathbb{A}} (d\mathbb{A})^T],$$

so that

$$(6.61) \quad \frac{\partial}{\partial d\mathbb{A}^1} \wedge \frac{\partial}{\partial d\mathbb{A}^2} S = -\overset{2}{\mathbb{A}} \overset{1}{\mathbb{A}} r^T + r^T \overset{1}{\mathbb{A}} \overset{2}{\mathbb{A}} + \overset{1}{\mathbb{A}} r^{T_1} \overset{2}{\mathbb{A}} - \overset{2}{\mathbb{A}} r^{T_1} \overset{1}{\mathbb{A}} = -\left(\overset{2}{\mathbb{A}} \overset{1}{\mathbb{A}} r - r \overset{1}{\mathbb{A}} \overset{2}{\mathbb{A}} - \overset{1}{\mathbb{A}} r^{T_1} \overset{2}{\mathbb{A}} + \overset{2}{\mathbb{A}} r^{T_1} \overset{1}{\mathbb{A}}\right),$$

as we wanted.

This is a rather interesting fact as it allows to pursue the same sort of approach based on the construction of the morphism groupoid for any Poisson algebra which admits R -matrix formulation. Indeed, we could define the $P_{\mathbb{A}}$ operator by (6.58) or (6.59), and construct the Lie algebroid for any given R . This would allow us to obtain the corresponding groupoid in which to find the generators of the braid group. This work will be carried out in subsequent publications. In particular it would be interesting to start from the chase of the exchange R -matrix in which all central elements are known [7], incidentally they are combinations of our rational central elements $b_d^{(I)}$ and their top right analogues.

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